NAME: ______________________ ID: ______________

SIGNATURE: ______________________

To get credit for a problem, you must show all of your reasoning and calculations. You may consult your books, notes, calculator, any materials from the CCLE site, the professor, or your TA. You may not collaborate or ask questions online. $\boxed{\textit{Box your final answer.}}$
If you cannot find a vector that you need for a later part of a problem, you may use the vector $\langle 1, 2, 3\rangle$.
If you cannot find a point that you need for a later part of a problem, you may use the point $(1, 1, 1)$.

Circle your section:

Section:	**Tuesday:**	**Thursday:**	**TA:**
	2A	2B	Alexander Johnson
	2C	2D	Francis White
	2E	2F	Jason Snyder

1. Consider the function $f(x, y) = x^2 + y^2$. Find the extrema of f subject to $3x + 4y = 5$ in two ways:

   (a) (10 points) Parameterize the constraint line & plug into f.

$f_{xx} = 2$

$f_{xy} = 0$

$f_{yx} = 0$

$f_{yy} = 2$

$3x + 4y = 5$

$3x + 4y - 5 = g(x,y) = 0$

$x(t) = t$

$r(t) = \langle x(t), y(t) \rangle$

$3t + 4y - 5 = 0 \qquad 4y = 5 - 3t$

$y(t) = \dfrac{5-3t}{4}$

$r(t) = \left\langle t, \dfrac{5-3t}{4} \right\rangle$

$f(t) = x(t)^2 + y(t)^2 = \dfrac{16}{16}t^2 + \left(\dfrac{5-3t}{4}\right)^2$

$= \dfrac{16t^2 + (5-3t)^2}{16} = \dfrac{16t^2 + 25 - 30t + 9t^2}{16}$

$= \dfrac{25t^2 - 30t + 25}{16} \qquad f'(t) = \dfrac{50t - 30}{16} = 0$

$50t - 30 = 0 \qquad t = \dfrac{30}{50} = \dfrac{3}{5}$

$$f''(t) = \frac{50}{16} > 0$$

$$\boxed{\begin{array}{l} f(x,y) \text{ on the line } 3x+4y=5 \text{ has a } \underline{\text{local min}} \\ \text{at } \left(3/5,\ \frac{5-3\cdot 3/5}{4}\right) = \left(3/5,\ 4/5\right) \end{array}}$$

$$\frac{\frac{25}{5} - \frac{9}{5}}{4} = \frac{16}{20} = \frac{4}{5}$$

(b) (10 points) Lagrange Multipliers

$$f(x,y) = x^2 + y^2$$

$$g(x,y) = 3x + 4y - 5 = 0$$

$$\nabla f = \lambda \nabla g$$

$$\nabla f = \langle f_x, f_y \rangle = \langle 2x, 2y \rangle$$

$$\nabla g = \langle 3, 4 \rangle$$

$$\langle 2x, 2y \rangle = \lambda \langle 3, 4 \rangle$$

$$\begin{aligned} 2x &= 3\lambda \\ 2y &= 4\lambda \end{aligned} \Rightarrow \begin{aligned} \frac{2x}{3} &= \lambda \\ \frac{2y}{4} &= \lambda \end{aligned} \Rightarrow \frac{2x}{3} = \frac{2y}{4}$$

$$x = \frac{3}{4} y$$

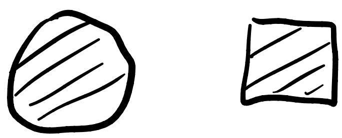
$$3x + 4y - 5 = 0 \Rightarrow \frac{9}{4} y + 4y = 5$$

$$\frac{25y}{4} = 5 \Rightarrow y = \frac{20}{25} = \frac{4}{5}$$

$$(3/5, 4/5)$$

$$x = \frac{3}{4} \cdot \frac{4}{5} = 3/5$$

(c) (5 points) Is the point you found a maximum or a minimum? Explain how you know, and explain why you weren't guaranteed to have a global maximum and minimum.



Not guaranteed global max or min because the constrained unbounded region

$g(x,y) = 0$

2. Let

$$f(x, y) = xe^{xy}.$$

(a) (10 points) Find the gradient of f.

$$f_x = 1 \cdot e^{xy} + x \cdot e^{xy} \cdot y = e^{xy}(1 + xy)$$

$$f_y = x \cdot e^{xy} \cdot x = x^2 e^{xy}$$

$$\nabla f = \langle e^{xy}(1+xy), x^2 e^{xy} \rangle$$

(b) (10 points) Find the linear approximation to $f(x, y)$ at the point $(1, 0)$.

Tangent plane

$$L(x,y) = z = f(1,0) + f_x(1,0)(x-1) + f_y(1,0)(y-0) + f_z(P)(z-c)$$

$$z = 1 \cdot e^0 + e^0(1+0)(x-1) + 1 \cdot e^0 (y)$$

$$= 1 + (x-1) + y = x + y$$

(c) (5 points) Use the linear approximation to estimate $f(1.1, -.1)$.

$$L(1.1, -.1) = 1.1 + (-.1) = 1$$

$$f(1.1, -.1) \approx 1$$

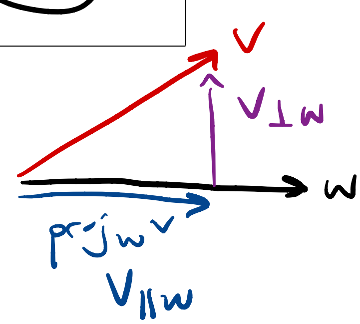
3. (2 points each) True/False! Circle the appropriate answer.

No justification is needed here.

(1) For any two vectors $\vec{v}$ and $\vec{u}$, $\vec{v}\cdot\vec{u} = \vec{u}\cdot\vec{v}$.	True	False	
(2) If $f(x,y)$ and $g(x,y)$ are continuous at (a,b), then the function $f(x,y)\cdot g(x,y)$ is continuous at (a,b)	True	False	
(3) For any two vectors $\vec{u}$ and $\vec{v}$, $$\vec{u}\times\vec{v} = \|\|\vec{u}\|\|\cdot\|\|\vec{v}\|\|\cdot\sin\theta,$$ where θ is the angle between $\vec{u}$ and $\vec{v}$.	True	False	$\|\|u\times v\|\| = \|\|u\|\|\,\|\|v\|\| \sin\theta$
(4) The x-component of a vector $\vec{v}$ always equals the dot product $\vec{v}\cdot\vec{\imath}$.	True	False	
(5) A continuous function on a closed but **not** bounded region in $\mathbb{R}^2$ cannot have an absolute maximum and minimum.	True	False	
(6) The gradient of f is ~~tangent~~ to the level curves.	True	False	normal
(7) The curvature of a curve in space can be negative.	True	False	
(8) Let $\vec{u}$, $\vec{v}$, and $\vec{w}$ be three vectors all of whose components are integers. Then $\vec{u}\cdot(\vec{v}\times\vec{w})$ is an integer.	True	False	
(9) For any two vectors $\vec{v}$ and $\vec{u}$, $\vec{v}\times\vec{u} = -\vec{u}\times\vec{v}$.	True	False	
* straight lines (10) If the limits of $f(x,y)$ along all * lines through a point (a,b) exist and agree, then the limit $\lim_{(x,y)\to(a,b)} f(x,y)$ exists.	True	False	
(11) The cross product of two unit vectors that are not parallel is always unit vector.	True	False	
~~(12) The set of points $\{(x,y) \mid 0 < x^2+y^2-9 \le 16\}$ is closed.~~	True	False	
(13) A continuous function on a closed and bounded region has an absolute maximum and minimum.	True	False	
(14) If a level curve intersects itself at a point so that there are two distinct tangent directions, then the point is a critical point	True	False	
(15) Given a vector $\vec{v}$ and a non-zero vector $\vec{w}$, we always have $\|\|\vec{v}_{\perp\vec{w}}\|\| \le \|\|\vec{v}\|\|$.	True	False	

$\|\|proj_w v\|\| \le \|\|v\|\|$

$v = \langle x, y, z\rangle \qquad i = \langle 1,0,0\rangle$

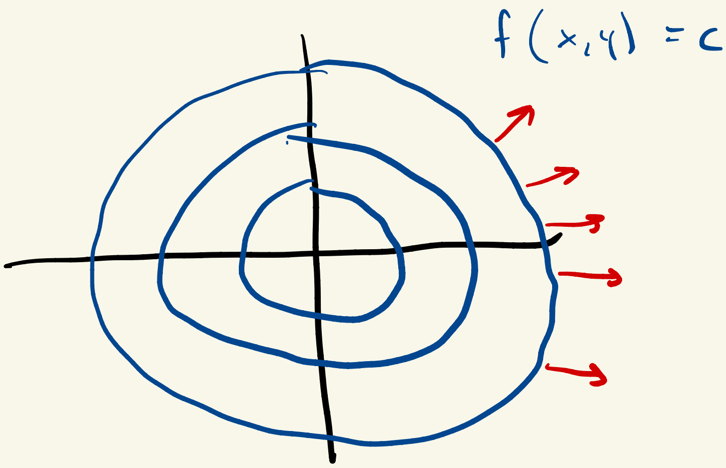


$v\cdot i = x\cdot 1 + y\cdot 0 + z\cdot 0 = x$

$v\cdot j = y \quad , \quad v\cdot k = z$

$x^2+y^2<1$

$x^2+y^2 \le 1$ closed



$$K(t) = \frac{\| r'(t) \times r''(t) \|}{\| r'(t) \|^3}$$

$$u \cdot (v \times w) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

volume of parallelepiped formed by u, v, w is $|u \cdot (v \times w)|$

$u = \langle u_1, u_2, u_3 \rangle$

$v = \langle v_1, v_2, v_3 \rangle$

$w = \langle w_1, w_2, w_3 \rangle$

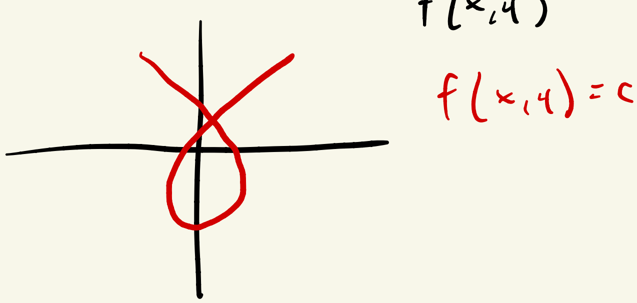
$$= u_1 (v_2 w_3 - w_2 v_3) - u_2 (v_1 w_3 - w_1 v_3) + u_3 (v_1 w_2 - w_1 v_2)$$

$$u \times (v \times w) \neq (u \times v) \times w$$

cross product is <u>NOT</u> associative

f(x,y)

(14)



$$\|u \times v\| = \|u\| \, \|v\| \sin\theta$$

$$\left.\begin{array}{l} \|u\| = 1 \\ \|v\| = 1 \end{array}\right\} \Rightarrow \quad \|u \times v\| = \sin\theta$$

$$0 \leq \sin\theta \leq 1$$

$u \times v$ may <u>not</u> be a unit vector

(14) Let $f(x,y)$ have a level curve $f(x,y) = c$ that intersects itself at a point (a,b). As the question states, we have two <u>non-parallel</u> tangent directions, i.e. two unit vectors v, w s.t. $D_v f(a,b) = 0$ and $D_w f(a,b) = 0$.

$$\Rightarrow \quad \begin{array}{l} \nabla f \cdot v = 0 \\ \nabla f \cdot w = 0 \end{array}$$

<u>But</u>, since v and w are <u>not</u> parallel, we have a vector $\nabla f(a,b)$ that is orthogonal to two non-parallel vectors. This can only happen if $\nabla f(a,b) = 0$.

4. Consider the hyperboloid of one sheet described by the equation

$z(x, y)$ $$x^2 + y^2 - z^2 = 4.$$ $F(x, y, z) = 0$

(a) (10 points) The equation defines z implicitly in terms of x and y. Find

$\frac{\partial z}{\partial x} = \frac{-F_x}{F_z}$

$F(x,y,z) = x^2 + y^2 - z^2 - 4 = 0$

$F_x = 2x$

$= \frac{-2x}{-2z} = \frac{x}{z}$

$F_y = 2y$

$F_z = -2z$

$\frac{\partial z}{\partial y} = \frac{-F_y}{F_z} = \frac{-2y}{-2z} = \frac{y}{z}$

(b) (10 points) Find the equation of the tangent plane at the point $(2, -2, -2)$.

Normal vector : $\nabla F = \langle F_x, F_y, F_z \rangle = \langle 2x, 2y, -2z \rangle$

$\nabla F(2, -2, -2) = \langle 4, -4, 4 \rangle$

$4x - 4y + 4z = d$

$4x - 4y + 4z = 8$

$4 \cdot 2 - 4(-2) + 4(-2) = d$

$x - y + z = 2$

$8 + 8 - 8 = d = 8$

$$z = f(x,y)$$

Tangent plane at (a,b) $\quad \big(a, b, f(a,b)\big)$:

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

$$F(x,y,z) = 0 \qquad (\text{Implicit})$$

Tangent plane at (a,b,c)

$$F_x(a,b,c)(x-a) + F_y(a,b,c)(y-b) + F_z(a,b,c)(z-c) = 0$$

Rederiving $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ formulas:

$$F(x,y,z) = 0$$

So $$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial x} \cdot \overset{1}{\cancel{\frac{\partial x}{\partial x}}} + \frac{\partial F}{\partial y} \cdot \overset{0}{\cancel{\frac{\partial y}{\partial x}}} + \frac{\partial F}{\partial z} \cdot \frac{\partial z}{\partial x} = 0$$

$$\Rightarrow \quad \frac{\partial z}{\partial x} = \frac{-F_x}{F_z}$$

$$\frac{\partial F}{\partial y} = \frac{\partial F}{\partial x} \cdot \overset{0}{\cancel{\frac{\partial x}{\partial y}}} + \frac{\partial F}{\partial y} \cdot \overset{1}{\cancel{\frac{\partial y}{\partial y}}} + \frac{\partial F}{\partial z} \cdot \frac{\partial z}{\partial y} = 0$$

$$\Rightarrow \quad \frac{\partial z}{\partial y} = \frac{-F_y}{F_z}$$

5. Consider the function
$$f(x, y) = x^4 - 2x^2 + y^4 - 8y^2.$$

(a) (20 points) Find the 9 critical points.

$f_x = 4x^3 - 4x = 0$ $\Rightarrow$ $x^3 - x = 0$ $\Rightarrow$ $x(x-1)(x+1) = 0$

$f_y = 4y^3 - 16y = 0$ $\Rightarrow$ $y^3 - 4y = 0$ $\Rightarrow$ $y(y-2)(y+1) = 0$

$x = 0, 1, -1$

$y = 0, 2, -1$

Crit pts: $(0,0)$, $(0,2)$, $(0,-1)$,
$(1,0)$, $(1,2)$, $(1,-1)$,
$(-1,0)$, $(-1,2)$, $(-1,-1)$

(b) (7 points) For this function, what is the discriminant D that plays a role in the 2nd derivative test?

$$f_{xx} = 12x^2 - 4$$

$$f_{xy} = 0$$

$$f_{yx} = 0$$

$$f_{yy} = 12y^2 - 16$$

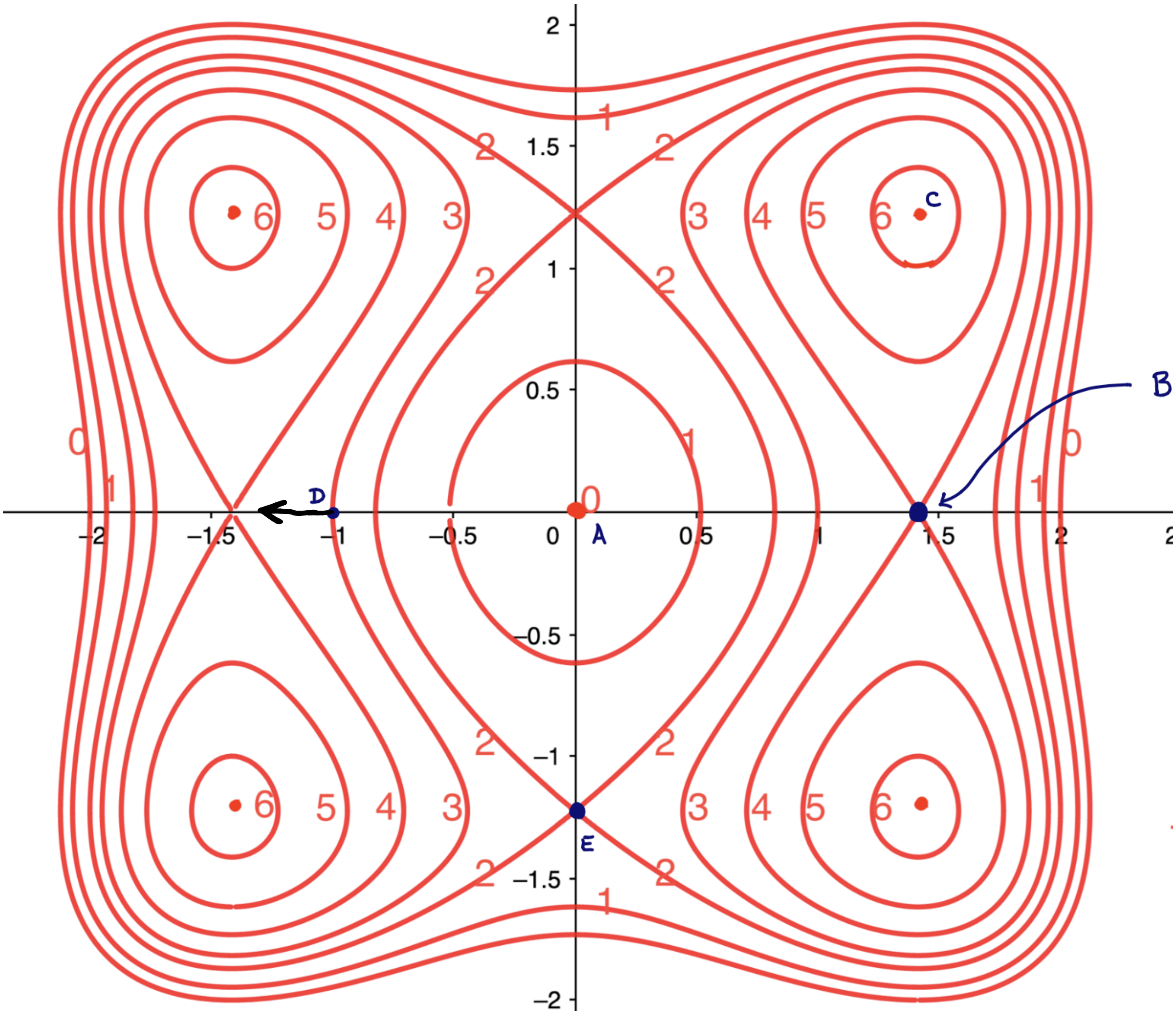
$$D = f_{xx}f_{yy} - f_{xy}^2 = (12x^2 - 4)(12y^2 - 16)$$

(c) (3 points each) Find one local maximum, one local minimum, and one saddle point for this function (you do not need to classify all of the critical points).

	D	f_{xx}	Type
$(0,0)$:	$-4 \cdot -16 = 64$	-4	max
$(1,0)$:	$8 \cdot -16 = -128$	8	saddle
$(-1,0)$:	-128	8	saddle
$(0,2)$:	$-4 \cdot 32 = -128$	-4	saddle
$(1,2)$:	$8 \cdot 32 = 256$	8	min
$(-1,2)$:	$8 \cdot 32 = 256$	8	min
$(0,-1)$:	$-4 \cdot -4 = 16$	-4	max
$(1,-1)$:	$8 \cdot -4 = -32$	8	saddle
$(-1,-1)$:	$8 \cdot -4 = -32$	8	saddle

6. Consider the contour plot of a function $f(x, y)$ below (where the contour labels are the larger numbers written in red next to the curves):

Gradient points in dir. of steepest increase of f



(a) (3 points each) The points labeled A, B, and C are critical points (and the only critical points within the next closest contour). Classify them as maxima, minima, or saddles. **Briefly** say how you know.

A : Min

B : saddle

C : Max

(b) (3 points) Draw the gradient at the point labeled D. No justification is needed.

(c) (3 points) What is the gradient at the point labeled E? **Briefly** explain how you know.

$\nabla f(E) = 0$ b/c E is a saddle point, so it's a critical pt, so $\nabla f = 0$

7. Consider the curve given by

$$\vec{r}(s) = \left\langle \frac{4s}{5}, 3\sin\left(\frac{s}{5}\right), -3\cos\left(\frac{s}{5}\right) \right\rangle.$$

must mean $\|r'(s)\| = 1$

(a) (5 points) Show that s is the arc-length parameter.

$$r'(s) = \left\langle \frac{4}{5}, 3\cos\left(\frac{s}{5}\right)\cdot\frac{1}{5}, 3\sin\left(\frac{s}{5}\right)\cdot\frac{1}{5} \right\rangle$$

$$\|r'(s)\| = \frac{1}{5}\sqrt{4^2 + \left(3\cos\left(\frac{s}{5}\right)\right)^2 + \left(3\sin\left(\frac{s}{5}\right)\right)^2}$$

$$= \frac{1}{5}\sqrt{16 + 9\cos^2 + 9\sin^2} = \frac{1}{5}\sqrt{16+9} =$$

$$\frac{1}{5}\sqrt{25} = 1$$

(b) (10 points) Find the unit normal and curvature as a function of s.

$$T(s) = \frac{r'(s)}{\|r'(s)\|} = \left\langle \frac{4}{5}, \frac{3}{5}\cos\left(\frac{s}{5}\right), \frac{3}{5}\sin\left(\frac{s}{5}\right) \right\rangle$$

$$T'(s) = \left\langle 0, -\frac{3}{5}\sin\left(\frac{s}{5}\right)\cdot\frac{1}{5}, \frac{3}{5}\cos\left(\frac{s}{5}\right)\cdot\frac{1}{5} \right\rangle$$

$$\|T'(s)\| = \frac{3}{25}\sqrt{\sin^2(s/5) + \cos^2(s/5)} = 3/25$$

$$N(s) = \frac{T'(s)}{\|T'(s)\|} = \left\langle 0, -\sin(s/5), \cos(s/5) \right\rangle$$

$$K(s) = \left\| \frac{dT}{ds} \right\| = 3/25$$

by definition

$$= \| T'(s) \|$$

8. Consider the function

$$f(x,y) = \frac{x^2}{x^2+y^2}$$

(a) (5 points) Where is f continuous and why?

not continuous when $x^2+y^2=0$

x^2 is cont. for all values of x & y

x^2+y^2 is cont. for all values of x & y

$\Rightarrow$ $f(x,y)$ is continuous everywhere except for where $x^2+y^2=0$, which is at $(x,y)=(0,0)$

$\mathbb{R}^2-(0,0)$

(b) (10 points) Is there a value a such that the function

$$\tilde{f}(x,y) = \begin{cases} \frac{x^2}{x^2+y^2} & (x,y) \neq (0,0) \\ a & (x,y) = (0,0) \end{cases}$$

is continuous? If yes, find it and show continuity. If no, show why not.

a would need to be $\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$

$x = r\cos\theta$

$y = r\sin\theta$

$$\frac{x^2}{x^2+y^2} = \frac{\cancel{r^2}\cos^2\theta}{\cancel{r^2}} = \cos^2\theta$$

$y=mx$

$$\frac{x^2}{x^2+m^2x^2} = \frac{x^2}{x^2(1+m^2)} = \frac{1}{1+m^2}$$

$\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$ along $y=mx$ is $\frac{1}{1+m^2}$, which depends on m, so it DNE

Since $\lim\limits_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$ DNE, $f(x,y)$ cannot be continuous at $(0,0)$ and no such a exists

9. A particle moves along the curve $y = x^3 - 4x$, shown below. Using x as the parameter, find

$r(x) = \langle x, x^3 - 4x \rangle$

(a) (5 points) The velocity of the particle

$v(x) = \langle 1, 3x^2 - 4 \rangle$

$r'(x) = \langle 1, 3x^2 - 4, 0 \rangle$

$\|r'(x)\|^3 = \sqrt{1 + (3x^2 - 4)} = (1 + (3x^2 - 4))^{3/2}$

$r'(x) \times r''(x) = \begin{vmatrix} i & j & K \\ 1 & 3x^2-4 & 0 \\ 0 & 6x & 0 \end{vmatrix}$

$= 6x\,K$

(b) (5 points) the acceleration of the particle

$a(x) = \langle 0, 6x \rangle$

$r''(x) = \langle 0, 6x, 0 \rangle$

(c) (5 points) the curvature of the curve as a function of x.

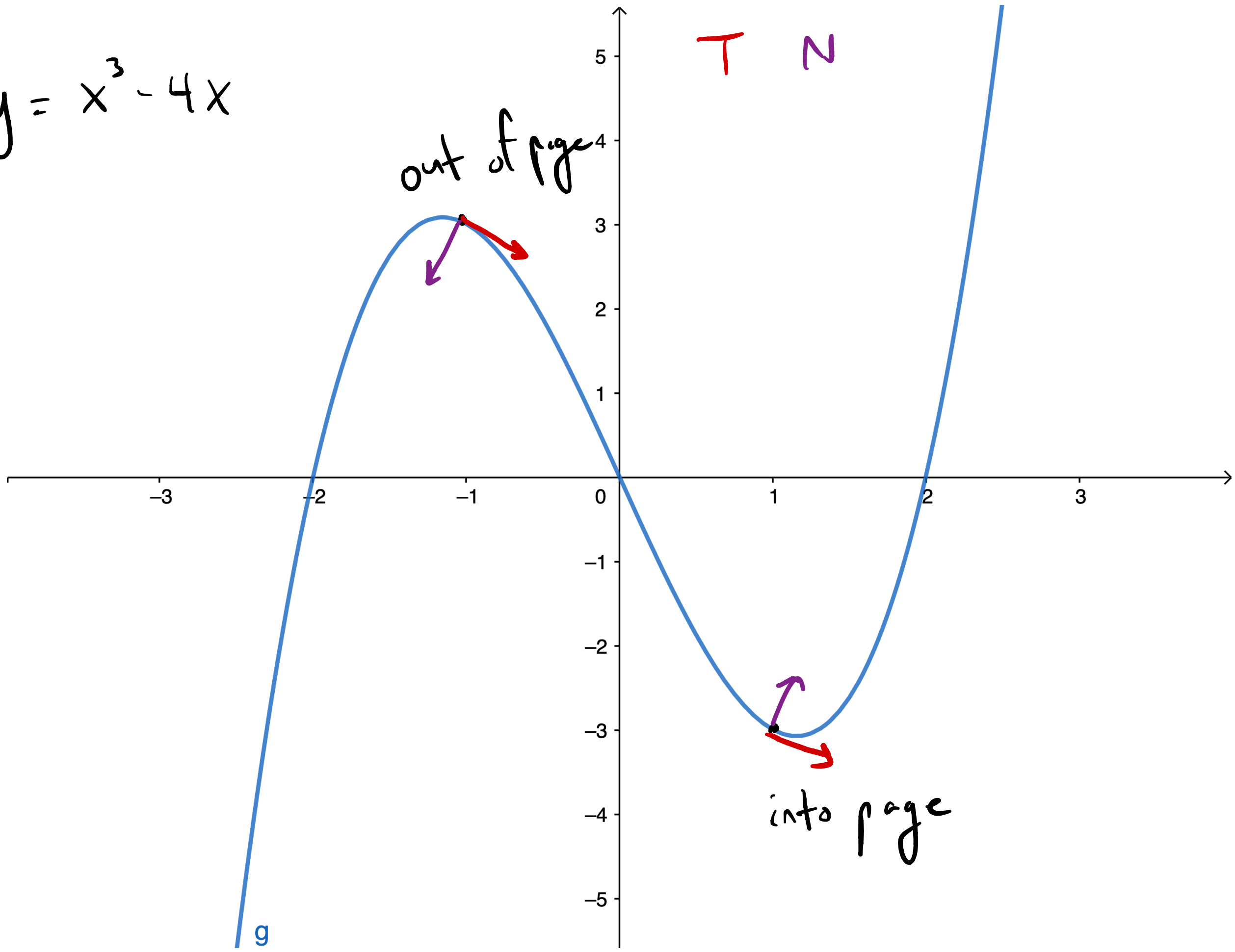
$$K(x) = \frac{|f''(x)|}{(1 + f'(x)^2)^{3/2}} = \frac{|6x|}{(1 + (3x^2 - 4)^2)^{3/2}}$$

//

$$K(x) = \frac{\|r'(x) \times r''(x)\|}{\|r'(x)\|^3}$$

(d) (5 points each) On the picture below, draw the unit tangent and unit normal vectors at $(-1,3)$ and $(1,-3)$. Next to each point, indicate if the binormal vector goes up out of the page or down into the page.

$y = x^3 - 4x$



$B = T \times N$

10. Let $\vec{u} = \langle 3, -2, a \rangle$, $\vec{v} = \langle 1, 0, 1 \rangle$, and $\vec{w} = \langle 0, 1, 1 \rangle$.

(a) (8 points) For what a is the vector triple product $\vec{u} \cdot (\vec{v} \times \vec{w}) = 0$?

$$u \cdot (v \times w) = \begin{vmatrix} 3 & -2 & a \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix}$$

$$= 3(0-1) + 2(1-0) + a(1) = 0$$

$$\Rightarrow \boxed{a = 1}$$

(b) (2 points) What does this mean for the parallelepiped spanned by $\vec{u}$, $\vec{v}$, and $\vec{w}$?

It's just a parallelogram.

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